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Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control of Multimodel Plants

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In this paper we consider the problem of minimizing a nominal $\mathcal{H}_2$ performance measure subject to robust stability constraints for systems having multiple operating points. Performance is measured in terms of a weighted sum of the individual $\mathcal{H}_2$ norms at each plant condition, whereas robust stability is enforced through $\mathcal{H}_\infty$-norm bounds. The problem is solved by considering a new time-domain scalar cost function $J_\infty(t_f)$ that incorporates the $\mathcal{H}_\infty$ constraints into a single cost function. Using the proposed formulation, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design problem becomes an unconstrained optimization problem that, for $t_f \to \infty$, recovers the original objective of minimizing the performance measure subject to the given $\mathcal{H}_\infty$ bounds. The resulting optimization problem is smooth, and hence standard gradient-based software can be applied. Moreover, this formulation allows for the synthesis of linear time-invariant controllers having a prespecified order and structure. The method is illustrated by the design of a single first-order controller for the pitch stability augmentation system of an F-15 aircraft operating at both subsonic and supersonic flight conditions.

I. Introduction

IN the past few years, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem has been the subject of much interest, since it allows the incorporation of robust stability into the linear quadratic Gaussian (LQG) framework. Robust $\mathcal{H}_2/\mathcal{H}_\infty$ performance still remains, to a large extent, an unsolved problem. An approach based upon parameter optimization methods can be found in Ref. 1, where necessary conditions for solving this problem using fixed-order controllers have been derived. Alternatively, a nominal performance with robust stability (NPRS) problem can be formulated, where the controller yields a desired performance level for the nominal system while guaranteeing stability for all possible plant perturbations. At the present time, no analytical solution is known for this problem. A related problem was formulated in Refs. 2 and 3, where the $\mathcal{H}_2$ cost is replaced by an upper bound. The dual problem to this modified NPRS problem has been solved in Ref. 4. In Ref. 5 it has been shown that the conditions derived in Refs. 3 and 4 are necessary and sufficient. These approaches are restricted to systems with identical disturbance inputs or identical criterion outputs and result in a set of coupled Riccati equations that is in general difficult to solve. For the same class of systems, the modified NPRS problem has been solved for static and dynamic state feedback[6,7] and dynamic full-order output feedback[8] controllers. There the problem was cast into a convex constrained optimization over a bounded set of matrices and solved using nondifferentiable constrained optimization techniques.

These approaches provide a solution to the modified problem. However, at this time little is known about the gap between the upper bound minimized in the modified problem and the actual $\mathcal{H}_2$ cost. Moreover, some recent numerical results[9] suggest that this gap may be significant. Very little work has been done concerning the original problem, which remains, to a large extent, still open. Recently, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem was solved in the single input/single output case using convex nondifferentiable optimization.[10] Although this approach provides an exact solution to the original problem, as pointed out in Ref. 10, it results in large-order controllers, necessitating some type of model reduction.

The approaches mentioned above provide an efficient way to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem when the controller is not restricted in its order or structure. However, modern control applications in aeronautics and astronautics often rely on modeling techniques based on finite element analysis and thus involve high-order plant models. Additional requirements such as fixed order and fixed structure have to be included in order to obtain a practical and implementable controller. One approach in this direction can be found in Refs. 11 and 12. However, in addition to a set of rather restrictive system assumptions (rank conditions as well as assumptions on system zeros), this approach requires an initial stabilizing controller guess that satisfies the $\mathcal{H}_\infty$ bound.

In this paper we address the NPRS $\mathcal{H}_2/\mathcal{H}_\infty$ problem using a time-domain-based penalty function approach. We formulate the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem as the problem of minimizing a time-domain cost function that explicitly incorporates the $\mathcal{H}_\infty$ bounds and is differentiable in all the variables involved. More importantly, the approach addresses the actual $\mathcal{H}_2$ cost rather than its upper bound and allows designers to impose limitation on the controller order as well as on its structure. As in previous approaches dealing with fixed-order/fixed-structure controllers, the resulting optimization problem is nonconvex. However, we believe that the approach proposed here has a number of advantages over previously proposed methods. In particular, 1) standard gradient-based optimization methods are directly applicable to this problem, 2) the approach incorporates multiple plant conditions and hence can simultaneously handle multiple operating points of the plant using static and dynamic controllers (fixed order, fixed structure, strictly proper or proper), 3) the overall cost function utilized for the approach is well defined even when the initial controller guess is not stabilizing, and finally, 4) the system assumptions are the least restrictive, requiring only the standard stabilizability and detectability conditions.

The paper is organized as follows. In Sec. II we formulate the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem for multiple-model plants and present some preliminary results. In Sec. III we show how to cast the $\mathcal{H}_\infty$ robustness constraints in the form of a smooth penalty function. In Sec. IV we combine this smooth penalty function with the $\mathcal{H}_2$ cost into a single objective function. In Sec. V a specific controller design algorithm is given. In Sec. VI we illustrate our approach with the design of a single first-order controller for the pitch stability augmentation

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system of an F-15 aircraft operating at both subsonic and supersonic flight conditions. Finally, in Sec. VII we summarize our results and provide some concluding thoughts.

II. Preliminaries

We will consider systems having n_p possible open-loop plant conditions indicated by the subscript op. Each plant condition ($i = 1, 2, \dots, n_p$) is represented by the following state-space model:

$$\Sigma^i_{2/\infty,\mathrm{op}}: \begin{cases} \dot{\bar{x}}^i(t) = \bar{A}^i\bar{x}^i(t) + \bar{B}^i_1 w^i_2(t) + \bar{B}^i_2 w^i_\infty(t) + \bar{B}^i_3 u^i(t) \\ z^i_2(t) = \bar{C}^i_1\bar{x}^i(t) + \bar{D}^i_{11} w^i_2(t) + \bar{D}^i_{12} w^i_\infty(t) + \bar{D}^i_{13} u^i(t) \\ z^i_\infty(t) = \bar{C}^i_2\bar{x}^i(t) + \bar{D}^i_{21} w^i_2(t) + \bar{D}^i_{22} w^i_\infty(t) + \bar{D}^i_{23} u^i(t) \\ y^i(t) = \bar{C}^i_3\bar{x}^i(t) + \bar{D}^i_{31} w^i_2(t) + \bar{D}^i_{32} w^i_\infty(t) \end{cases} \tag{1}$$

Here the superscript i indicates the ith plant, and the vectors $\bar{x}^i(t), u^i(t), y^i(t)$ represent the corresponding states, control action, and outputs available to the controller. Finally, the pairs $(w^i_2(t), z^i_2(t))$ and $(w^i_\infty(t), z^i_\infty(t))$ represent the disturbance input and performance output corresponding to the ith $\mathcal{H}_2$ performance measure and ith $\mathcal{H}_\infty$ constraint, respectively. The exogenous disturbances $w^i_2(t)$ are assumed to be white-noise signals with unit spectral density, thus representing the limiting case of disturbances with bounded spectrum (see, e.g., Ref. 4). For a given plant condition, model uncertainty is handled in the usual manner by lumping all uncertainties into a stable, norm-bounded $\Delta^i(s)$ block with $\|\Delta^i(s)\|_\infty \le 1/\gamma^i$ such that $w^i_\infty(s) = \Delta^i(s)z^i_\infty(s)$. Direct feedthrough terms from $u^i(t)$ to $y^i(t)$ in Eq. (1) can be incorporated as long as the corresponding feedthrough matrices $\bar{D}^i_{33}$ are the same for every plant condition and the final controller design is well posed.[13]

In this formulation individual plant conditions can be used to represent different operating conditions of the same plant. Alternatively, the n_p open-loop systems $\Sigma^i_{2/\infty,\mathrm{op}}$ may be used to account for different types of disturbances encountered in the design specifications (i.e., step responses, random responses). In each of the n_p plants, disturbance filters of different dynamic degree are allowed since our formulation does not place any restriction on the dimension of the individual state vectors $\bar{x}^i$.

For technical reasons we will impose the following assumptions on the set of open-loop plants for all $i = 1, 2, \dots, n_p$:

A1) $(\bar{A}^i, \bar{B}^i_3)$ are stabilizable pairs.
A2) $(\bar{A}^i, \bar{C}^i_3)$ are detectable pairs.
A3) $\dim(u^i) = n_{u^i} = n_u$ and $\dim(y^i) = n_{y^i} = n_y$.

Assumptions A1 and A2 are necessary for the existence of a controller that stabilizes all plant conditions simultaneously. That is, a controller must be able to detect unstable modes through $y^i(t)$ in any of the n_p open-loop plants and stabilize these modes via the control $u^i(t)$. The number of controller inputs and outputs must be the same for all $\Sigma^i_{2/\infty,\mathrm{op}}$ since we consider a single control law that is applied to all the plant conditions. This necessity is reflected in assumption A3.

We will search for finite-dimensional linear time-invariant controllers $C(s)$ having state-space realizations of the form

$$C: \begin{cases} \dot{x}_c(t) = A_c x_c(t) + B_c y^i(t) \\ u^i(t) = C_c x_c(t) + D_c y^i(t) \end{cases} \tag{2}$$

where the controller order n_c is prespecified (and not necessarily equal to the order of the plant). The structure of the matrices A_c, B_c, C_c, and D_c can be arbitrary and is assumed to be specified by the designer. For convenience, we adopt the following compact representation C_o of a controller $C(s)$ having the state-space realization (2):

$$C_o = \begin{pmatrix} D_c & C_c \\ B_c & A_c \end{pmatrix} \tag{3}$$

In order to obtain well-posed problems, we will make the following assumptions on the controller:

A4) $\bar{D}^i_{11} + \bar{D}^i_{13}D_c\bar{D}^i_{31} = 0$ for all $i = 1, 2, \dots, n_p$.
A5) $D_c\bar{D}^i_{32} = 0$ and/or $\bar{D}^i_{23}D_c = 0$ for all $i = 1, 2, \dots, n_p$.

Assumption A4 is a necessary condition in order to obtain a finite $\mathcal{H}_2$ norm for the closed-loop transfer functions from $w^i_2(s)$ to $z^i_2(s)$. If this assumption is not trivially satisfied by the structural formulation of the problem (i.e., a strictly proper controller design with $\bar{D}^i_{11} = 0$ for all $i = 1, 2, \dots, n_p$), then the design algorithm has to incorporate these constraints as additional (linear) equality constraints on the elements in D_c. This case is not considered here. Finally, assumption A5 is a technical assumption that greatly simplifies the further development and in particular the gradient expressions. For the general case without assumption A5 the reader is referred to Ref. 14.

Given a controller $C(s)$ with the state-space realization (2), the closed-loop plants (identified by the subscript cl) are given by

$$\Sigma^i_{2/\infty,\mathrm{cl}}(C_o): \begin{cases} \dot{x}^i(t) = A^i x^i(t) + B^i_1 w^i_2(t) + B^i_2 w^i_\infty(t) \\ z^i_2(t) = C^i_1 x^i(t) + D^i_{12} w^i_\infty(t) \\ z^i_\infty(t) = C^i_2 x^i(t) + D^i_{21} w^i_2(t) + D^i_{22} w^i_\infty(t) \end{cases} \tag{4}$$

Let $\Sigma^i_{2,\mathrm{cl}}(C_o)$ denote the subsystem of $\Sigma^i_{2/\infty,\mathrm{cl}}(C_o)$ from $w^i_2(s)$ to $z^i_2(s)$ with $w^i_\infty(s) = 0$,

$$\Sigma^i_{2,\mathrm{cl}}(C_o): \begin{cases} \dot{x}^i_2(t) = A^i x^i_2(t) + B^i_1 w^i_2(t) \\ z^i_2(t) = C^i_1 x^i_2(t) \end{cases} \tag{5}$$

and $\Sigma^i_{\infty,\mathrm{cl}}(C_o)$ the subsystem from $w^i_\infty(s)$ to $z^i_\infty(s)$ with $w^i_2(s) = 0$,

$$\Sigma^i_{\infty,\mathrm{cl}}(C_o): \begin{cases} \dot{x}^i_\infty(t) = A^i x^i_\infty(t) + B^i_2 w^i_\infty(t) \\ z^i_\infty(t) = C^i_2 x^i_\infty(t) + D^i_{22} w^i_\infty(t) \end{cases} \tag{6}$$

With $D_c\bar{D}^i_{32} = 0$ and/or $\bar{D}^i_{23}D_c = 0$ (assumption A5), it can be easily verified that $D^i_{22} = \bar{D}^i_{22}$ for all $i = 1, 2, \dots, n_p$, and hence the direct feedthrough matrices from $w^i_2(t)$ to $z^i_2(t)$ in Eq. (6) are not dependent on C_o. However, all other closed-loop matrices in Eqs. (4–6) are functions of the controller parameterization C_o. For notational simplicity, we will not indicate this dependency explicitly. Let $T^i_2(C_o, s)$ and $T^i_\infty(C_o, s)$ represent the transfer functions corresponding to the closed-loop systems $\Sigma^i_{2,\mathrm{cl}}(C_o)$ and $\Sigma^i_{\infty,\mathrm{cl}}(C_o)$, respectively. With this notation we can now precisely define the multimodel $\mathcal{H}_2/\mathcal{H}_\infty$ control problem as follows:

Problem 1 (mixed $\mathcal{H}_2/\mathcal{H}_\infty$ multimodel control). Find a stabilizing controller C_o such that the performance criterion $J_2(C_o)$ is minimized, where

$$J_2(C_o) = \min_{C_o} \lim_{t_f\to\infty} J_2(C_o, t_f) \tag{7}$$

$$J_2(C_o, t_f) = \sum_{i=1}^{n_p} \alpha^i J^i_2(C_o, t_f) \tag{8}$$

$$J^i_2(C_o, t_f) = \mathcal{E}\left[z^{iT}_2(t_f)z^i_2(t_f)\right] \tag{9}$$

$$\lim_{t_f\to\infty} J^i_2(C_o, t_f) = \lim_{t_f\to\infty} \mathcal{E}\left[z^{iT}_2(t_f)z^i_2(t_f)\right] = \left\|T^i_2(C_o, s)\right\|^2_2 \tag{10}$$

subject to the robust stability constraints

$$\left\|T^i_\infty(C_o, s)\right\|_\infty < \gamma^i \qquad i = 1, 2, \dots, n_p. \tag{11}$$

Here $z^i_2(t)$ denotes the responses of $\Sigma^i_{2,\mathrm{cl}}(C_o)$ to the random disturbances $w_2(t)$ of unit power spectra, $\mathcal{E}$ represents the expectation operator, the scalars α^i are n_p weighting factors chosen by the designer, and $1/\gamma^i$ is the upper bound on the unstructured uncertainty $\Delta^i(s)$ at the ith plant condition.

This is a constrained optimization problem where the $\mathcal{H}_\infty$-bound constraints can be expressed in terms of $\mathcal{H}_\infty$ matrix algebraic Riccati equations $\mathrm{ARE}^i(C_o, X^i) = 0$ or $\mathcal{H}_\infty$ matrix algebraic Riccati inequalities $\mathrm{ARI}^i(C_o, X^i) < 0$.

Lemma 1. Consider a linear, stable, time-invariant system Σ,

$$\Sigma: \begin{cases} \dot{x}(t) = Ax(t) + Bw(t) \\ z(t) = Cx(t) + Dw(t) \end{cases} \tag{12}$$

with transfer function $T(s)$ and $\gamma > \bar{\sigma}(D)$. Let $R = \gamma^2 I - D^T D$ and $S = \gamma^2 I - DD^T$. Then the following statements are equivalent (see, e.g., Ref. 15):

1) $\|T(s)\|_\infty < \gamma$.

2) Algebraic Riccati equation (ARE): Assume (A, B) to be controllable and (C, A) to be observable. Then the matrix equation

$$[A^T + C^T D R^{-1} B^T] Z + Z [A^T + C^T D R^{-1} B^T]^T + Z B R^{-1} B^T Z + \gamma^2 C^T S^{-1} C = 0 \tag{13}$$

has a unique, real, symmetric positive-definite solution Z such that $A + BR^{-1}[D^T C_o + B^T Z]$ is asymptotically stable.

3) Algebraic Riccati inequality (ARI): There exists a symmetric positive-definite matrix X such that[7,15]

$$[A^T + C^T D R^{-1} B^T] X + X [A^T + C^T D R^{-1} B^T]^T + X B R^{-1} B^T X + \gamma^2 C^T S^{-1} C < 0 \tag{14}$$

This lemma provides analytical tools to determine whether or not a given system satisfies an $\mathcal{H}_\infty$ constraint. The ARE (13) is the basis for most existing approaches to $\mathcal{H}_\infty$ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems. In this paper we will pursue a different approach based upon the use of the ARI (14) to define a time-domain function $J_\infty^i[\mathrm{ARI}^i(C_o, X^i), t_f]$ representing the ith $\mathcal{H}_\infty$ constraint. In this formulation, the controller matrix C_o and the n_p matrices X^i become independent optimization variables in a gradient-based minimization algorithm that attempts to achieve $\mathrm{ARI}^i(C_o, X^i) < 0$ for all $i = 1, 2, \ldots, n_p$ while minimizing the performance cost. This formulation offers the advantage over other methods used to enforce matrix inequality constraints (such as interior point methods[16]) of not requiring an initial guess already satisfying the $\mathcal{H}_\infty$ constraint.

III. Penalty Function for $\mathcal{H}_\infty$ Constraints

Given n_p systems $\Sigma_{\infty,\mathrm{cl}}^i(C_o)$ as in Eq. (6), let χ denote an n_p-tuple of symmetric positive-definite matrices, i.e.,

$$\chi \doteq \{X^1, X^2, \ldots, X^i, \ldots, X^{n_p} : X^i \in R^{n_{x^i} \times n_{x^i}}, X^i = X^{iT}, X^i > 0\} \tag{15}$$

where n_{x^i} is the order of the ith closed-loop system. Now consider the following penalty function:

$$J_\infty(C_o, \chi, t_f) = \sum_{i=1}^{n_p} J_\infty^i(C_o, X^i, t_f) \tag{16}$$

where

$$J_\infty^i(C_o, X^i, t_f) = \mathrm{Tr}\{\exp[\mathrm{ARI}^i(C_o, X^i) t_f]\} \tag{17}$$

$$\mathrm{ARI}^i(C_o, X^i) = [A^{iT} + C_2^{iT} D_{22}^i (R^i)^{-1} B_2^{iT}] X^i + X^i [A^{iT} + C_2^{iT} D_{22}^i (R^i)^{-1} B_2^{iT}]^T + X^i B_2^i (R^i)^{-1} B_2^{iT} X^i + \gamma^{i2} C_2^{iT} (S^i)^{-1} C_2^i \tag{18}$$

with $R^i = [(\gamma^i)^2 I - D_{22}^{iT} D_{22}^i]$ and $S^i = [(\gamma^i)^2 I - D_{22}^i D_{22}^{iT}]$. In the sequel we will show that the $\mathcal{H}_\infty$ constraints (11) can be reformulated in this context by considering the following minimization problem:

$$J_\infty(C_o^*, \chi^*) = \min_{C_o, \chi} \lim_{t_f \to \infty} J_\infty(C_o, \chi, t_f) \tag{19}$$

The key properties associated with the penalty function (16) and the optimization problem (19) are expressed in the following theorem.

Theorem 1. Consider a controller $\hat{C}_o$ and the corresponding n_p closed-loop plants $\Sigma_{\infty,\mathrm{cl}}^i(\hat{C}_o)$. Assume $\gamma^i > \bar{\sigma}(D_{22}^i)$ for all $i = 1, 2, \ldots, n_p$. Then the following is true:

$$\left\| T_\infty^i(\hat{C}_o, s) \right\|_\infty < \gamma^i \Leftrightarrow \min_{\chi} \lim_{t_f \to \infty} J_\infty(\hat{C}_o, \chi, t_f) = 0 \qquad \forall i \quad 1 \le i \le n_p \tag{20}$$

Furthermore, in this case the controller $\hat{C}_o$ internally stabilizes all plant conditions simultaneously. Conversely, the controller $\hat{C}_o$ violates at least one of the $n_p \mathcal{H}_\infty$ constraints if and only if $\min_\chi \lim_{t_f \to \infty} J_\infty(\hat{C}_o, \chi, t_f)$ is unbounded. Namely,

$$\left\| T_\infty^j(\hat{C}_o, s) \right\|_\infty > \gamma^j \Leftrightarrow \min_{\chi} \lim_{t_f \to \infty} J_\infty(\hat{C}_o, \chi, t_f) \to \infty \tag{21}$$

for at least one $j \, (1 \le j \le n_p)$.

Proof. Assume that $\hat{C}_o$ satisfies $\|T_\infty^i(\hat{C}_o, s)\|_\infty < \gamma^i$ for $1 \le i \le n_p$. Then there exists a n_p-tuple χ such that $\mathrm{ARI}^i(\hat{C}_o, X^i) < 0$ for all $i = 1, 2, \ldots, n_p$. This together with the symmetry of $\mathrm{ARI}^i(\hat{C}_o, X^i)$ implies that all its eigenvalues are real and strictly negative. Hence, for all $i = 1, 2, \ldots, n_p$,

$$\lim_{t_f \to \infty} \exp[\mathrm{ARI}^i(C_o, X^i) t_f] = 0$$

The right-hand side of Eq. (20) now follows immediately. Conversely, if $\min_\chi \lim_{t_f \to \infty} J_\infty(\hat{C}_o, \chi, t_f)$ is zero, then there exists an n_p-tuple χ of symmetric positive-definite matrices X^i such that all n_p matrix expressions $\mathrm{ARI}^i(\hat{C}_o, X^i)$ are negative definite. This in turn implies that all $\mathcal{H}_\infty$ constraints are satisfied. Furthermore, in this case internal closed-loop stability for all n_p plant conditions follows directly from a standard Lyapunov argument (see Ref. 14). This completes the proof of the first part of the theorem. The second part of the proof follows immediately from these considerations. □

Remark 1. The scalar cost $J_\infty^i(C_o, \chi, t_f)$ can be interpreted as the trace of the transition matrix of a system $\dot{e}(t) = \mathrm{ARI}^i(C_o, X^i) e(t)$ so that enforcing the $\mathcal{H}_\infty$ constraints can be viewed as the problem of simultaneously stabilizing n_p plants. This justifies the usage of the term time-domain penalty function. It is obvious that, in the limit as $t_f \to \infty$, the penalty function becomes a barrier function as $J_\infty(C_o, \chi, t_f)$ will become unbounded if one or more of the matrix inequality constraints are violated.

Remark 2. Given any finite t_f, $J_\infty(C_o, \chi, t_f)$ is continuous and differentiable with respect to C_o and all X^i. Explicit gradient expressions can be found in the Appendix and in Ref. 14. This property invites the use of gradient-based optimization algorithms to solve both multimodel $\mathcal{H}_\infty$ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problems.

Remark 3. Given a convex Hermitian matrix $Q(x)$ [i.e., $Q(\alpha x_1 + (1-\alpha) x_2) \le \alpha Q(x_1) + (1-\alpha) Q(x_2)$], it can be shown (see Ref. 17) that $\mathrm{Tr}\{\exp[Q(x) t_f]\}$ is convex for any given t_f. Hence, the proposed approach allows for converting Hermitian matrix-valued constraints into scalar differentiable functions retaining their convexity properties. This is important as many design objectives can be expressed in terms of convex (and often linear) matrix inequalities (see Ref. 16). Furthermore, since arbitrary Hermitian matrix constraints can be incorporated into a scalar cost using the penalty function, assumption A5 may be removed by enforcing the additional matrix inequalities $(D_{22}^i)^T D_{22}^i - (\gamma^i)^2 I < 0, 1 \le i \le n_p$, where D_{22}^i is now a function of the controller matrix C_o. However, in this case $\mathrm{ARI}^i(\hat{C}_o, X^i)$ is not necessarily differentiable in C_o, and alternative inequality representations of the $\mathcal{H}_\infty$ constraints must be used (see Ref. 14).

IV. Cost Function for Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Design

The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design problem can be solved using nondifferentiable constrained optimization methods. However, convergence to a solution is slow.[10] For improved numerical efficiency, we will use penalty function methods (see e.g., Ref. 18) to obtain an equivalent unconstrained problem that is differentiable.

Problem 2. Find a stabilizing controller C_o such that the performance criterion

$$J_{2/\infty}(C_o, \chi, t_f) = c_2 J_2(C_o, t_f) + J_\infty(C_o, \chi, t_f) \tag{22}$$

is minimized, where c_2 is a scaling factor.

Remark 4. Consider the limit case of Eq. (22) as $t_f \to \infty$, i.e.,

$$J_{2/\infty}^*(C_o^*, \chi^*) = \min_{C_o, \chi} \lim_{t_f \to \infty} J_{2/\infty}(C_o, \chi, t_f) \tag{23}$$

If this problem has a solution C_o^* such that $\text{ARI}^i(C_o^*, (X^i)^*) < 0$ and $(X^i)^* > 0$ for $i = 1, 2, \ldots, n_p$, this solution also solves Problem 1. This follows from Theorem 1 and the fact that $\text{ARI}^i(C_o^*, (X^i)^*) < 0$ and $(X^i)^* > 0$ for $i = 1, 2, \ldots, n_p$ if and only if $\|T_\infty^i(\hat{C}_o, s)\|_\infty < \gamma^i$, $i = 1, 2, \ldots, n_p$. Hence,

$$\lim_{t_f \to \infty} J_\infty(C_o^*, \chi^*, t_f) = 0 \tag{24}$$

Also note that from Theorem 1 it follows that the controller C_o^* simultaneously stabilizes all plant conditions. Thus, stability of all closed-loop matrices A^i is a natural result of this formulation and does not need to be explicitly enforced by means of additional constraints. Stability together with assumption A5 guarantees that the n_p $\mathcal{H}_2$ norms $\|T_2^i(C_o^*)\|_2$ are well defined and that

$$\lim_{t_f \to \infty} J_{2/\infty}(C_o^*, \chi^*, t_f) < \infty \tag{25}$$

Moreover, in this case

$$\lim_{t_f \to \infty} J_{2/\infty}(C_o^*, \chi^*, t_f) = \sum_{i=1}^{n_p} \alpha^i \left\| T_2^i(C_o^*) \right\|_2^2 \tag{26}$$

which is simply the $\mathcal{H}_2$ performance cost. On the other hand, if one or more of the $\mathcal{H}_\infty$ constraints are not satisfied or the controller C_o^* does not internally stabilize one or more of the plants, then $J_2(C_o^*, t_f)$, $J_\infty(C_o^*, \chi^*, t_f)$, or both will be unbounded in the limit as $t_f \to \infty$. Hence the overall cost function will become unbounded as well. Thus in this limit, $J_{2/\infty}(C_o^*, \chi^*, t_f \to \infty)$ is finite if and only if C_o^* and χ^* can be found such that all closed-loop plant conditions are internally stable and all n_p ARI constraints are satisfied.

It is important to note that the function $J_{2/\infty}(C_o, \chi, t_f)$ is convex neither in C_o nor in χ. Convexity can be achieved in the cases of static and dynamic state feedback for the pure $\mathcal{H}_\infty$ problem only (see, e.g., Ref. 7). For the general case of controllers with fixed structure and/or order, convexity results have not yet been established.

V. Proposed Design Algorithm

At this point we assume that the designer has chosen a set of n_p weighting factors α^i for the $\mathcal{H}_2$ performance cost and n_p $\mathcal{H}_\infty$ bounds γ^i [from Lemma 1 and assumption A5 it follows that these bounds must satisfy the constraints $\gamma^i > \bar{\sigma}(\bar{D}_{22}^i)$ in order for the problem to be solvable]. Additionally, an initial controller guess C_o of the desired structure and order and an initial guess for the set χ are required. As we will explain in detail later, this initial controller is not required to simultaneously stabilize the n_p plants. This is relevant since the problem of simultaneously stabilizing several plants is difficult. While it is well known that two plants P_1 and P_2 can be simultaneously stabilized if and only if their difference $P = P_1 - P_2$ is strongly stabilizable (see Theorem 4, Chapter 5 of Ref. 19), it has been recently shown[20] that the problem of checking whether or not there exists a controller that simultaneously stabilizes three or more plants is undecidable.

With these initial data the optimization problem 2 can be solved using two different approaches:

1) $J_{2/\infty}(C_o, \chi, t_f)$ can be minimized using a penalty function approach. An initial horizon $t_{f,0}$ is selected and the corresponding minimization problem (23) is solved. Since in this approach the $\mathcal{H}_2$ cost is computed using the finite-time cost functional (8), which is well defined even for destabilizing controllers, the initial controller is not required to simultaneously stabilize all plants. Gradients for this type of $\mathcal{H}_2$ performance have been derived in Ref. 21. Once a solution of Eq. (23) has converged for $t_{f,j-1}$, we increase the finite time to $t_{f,j}$. This process terminates when the largest implementable $t_{f,j}$ has been reached or when the algorithm has converged to a steady-state value in terms of $t_{f,j}$. In the latter case the controller parameters do not change significantly (as a function of t_f) and all the $\mathcal{H}_\infty$ constraints have been satisfied. The factor c_2 in $J_{2/\infty}(C_o^*, \chi^*, t_f)$ is used to properly scale the optimization problem so that the $\mathcal{H}_2$ cost dominates the overall cost once all ARI constraints have been satisfied.

2) On the other hand, given the same initial data as above, we can first find a controller that satisfies all the $\mathcal{H}_\infty$ constraints, disregarding the performance objective. After such a controller has been found, we optimize $J_{2/\infty}(C_o, \chi, t_f)$ until the performance cost cannot be improved upon. Such an algorithm may be described as follows:

Phase 1: Computation of $\mathcal{H}_\infty$ controller. Set $c_2 = 0$ and iteratively optimize on $J_{2/\infty}(C_o, \chi, t_f)$ to find a feasible controller $C_{o,0}$ that satisfies all n_p $\mathcal{H}_\infty$ constraints and a set χ_0 such that all the ARI-related matrix constraints are satisfied. It follows that

$$\lim_{t_f \to \infty} J_\infty(C_{o,0}, \chi_0, t_f) = 0 \tag{27}$$

If a feasible controller cannot be found, then the algorithm terminates at this point; otherwise this controller (which from Theorem 1 simultaneously stabilizes all plants) can then be used as an initial guess for the second phase of the algorithm, where the performance cost is taken into account. This second phase solves the optimization problem $\min_{C_o,\chi} J_{2/\infty}(C_o, \chi, t_{f,0})$ for a large $t_{f,0}$, as follows:

Phase 2: Computation of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller. At the jth iteration do the following: Select $t_{f,j} > 1$ and the scaling factor $c_{2,j}$ such that

$$J_\infty(C_{o,j-1}, \chi_{j-1}, t_{f,j}) \ll 1 \tag{28}$$

$$c_{2,j} J_2(C_{o,j-1}, \infty) = c_{2,j} \sum_{i=1}^{n_p} \alpha^i \left\| T_2^i(C_{o,j-1}) \right\|_2^2$$

$$\gg J_\infty(C_{o,j-1}, \chi_{j-1}, t_{f,j}) \tag{29}$$

$$J_{2/\infty}(C_{o,j-1}, \chi_{j-1}, t_{f,j}) < 1 \tag{30}$$

For these parameter settings solve the unconstrained minimization problem

$$\min_{C_o,\chi} J_{2/\infty}(C_o, \chi, t_{f,j}) \tag{31}$$

to get $C_{o,j}$ and the set χ_j. If

$$\left| \sum_{i=1}^{n_p} \alpha^i \left\| T_2^i(C_{o,j-1}) \right\|_2^2 - \sum_{i=1}^{n_p} \alpha^i \left\| T_2^i(C_{o,j}) \right\|_2^2 \right| < \varepsilon \tag{32}$$

for some prespecified ε, then stop. Otherwise let $j = j + 1$ and repeat phase 2 of this algorithm.

The selection of $t_{f,j}$ and $c_{2,j}$ at the jth iteration guarantee that the $\mathcal{H}_2$ performance cost dominates the $\mathcal{H}_\infty$ cost during the optimization (31). Furthermore, inequality (30) implies that the function value $J_{2/\infty}(C_o, \chi, t_{f,j})$ remains strictly smaller than 1. In turn, this fact, together with $t_{f,j} > 1$, implies that $J_\infty(C_o, \chi, t_{f,j}) < 1$ for all C_o. From Theorem 1 it is easily verified that this implies that throughout phase 2 of the algorithm the following two conditions hold: 1) the controller C_o remains stabilizing for all plant conditions and 2) all n_p constraints $\text{ARI}^i(C_o, X^i) < 0$ are satisfied. Hence, in this formulation $J_\infty(C_o, \chi, t_f)$ acts as a barrier function enforcing the $\mathcal{H}_\infty$ constraints and stability during the entire optimization process in phase 2. Furthermore, these properties allow the computation of the $\mathcal{H}_2$ cost and its gradients via observability or controllability gramians, which require solving only Lyapunov equations. Corresponding gradients for steady-state $\mathcal{H}_2$ costs are readily available (see, e.g., Refs. 21 and 14). The algorithm terminates when the $\mathcal{H}_2$ cost cannot be reduced any further.

Note that both approaches are based upon the use of a finite horizon. This allows for avoiding numerical overflow problems arising from a destabilizing initial controller guess or a poorly chosen initial set χ (resulting in large eigenvalues for the corresponding ARIs) by choosing a small $t_{f,0}$. The numerical implementation utilizes the optimization toolbox of MATLAB as well as the software package SANDY.[21] In the example below, we applied the barrier function approach (second algorithm).

Remark 5. The number of optimization variables n_{con} associated with the controller parameterization C_o is fixed by the choice of the controller structure and order. Due to the symmetry requirement for X^i, we do not have to use $(n_{x^i} + n_c)^2$ optimization variables to represent X^i in the optimization process. Using a Cholesky factorization of $X^i = Q^{iT} Q^i$, we optimize over a set of n_p upper triangular matrices Q^i of appropriate dimensions rather than over χ. Hence the assumptions $X^i = X^{iT} \geq 0$ are explicitly accounted for. Strict inequality, that is, $X^i > 0$, is incorporated by constraining the diagonal elements $[Q^i]_{jj}$ of the Cholesky factors Q^i to be strictly positive. Hence the additional constraints

$$[Q^i]_{jj} \geq \frac{\varepsilon_o}{t_f} \qquad \varepsilon_o > 0, \qquad 1 \leq j \leq n_{x^i}, \qquad 1 \leq i \leq n_p \tag{33}$$

will guarantee the required positive definiteness of all X^i for any finite t_f. Strict inequality of the ARI constraints can also be enforced in a similar manner. With this formulation the overall number n_{var} of optimization variables is given by

$$n_{\text{var}} = n_{\text{con}} + \frac{1}{2}\sum_{i=1}^{n_p}(n_{\bar{x}^i} + n_c)(n_{\bar{x}^i} + n_c + 1) \tag{34}$$

where $n_{\bar{x}^i}$ is the dimension of the ith open-loop plant.

VI. Illustrative Example

We illustrate our approach by designing a first-order controller for the fourth-order longitudinal dynamic model of an F-15 aircraft at two operating points.[23] The first plant condition represents a subsonic flight condition whereas the second operating condition is supersonic. A state-space realization of the linearized plant model is given by

$$\begin{bmatrix} \dot{V}(t) \\ \dot{\alpha}(t) \\ \dot{q}(t) \\ \dot{\theta}(t) \end{bmatrix} = \bar{A}^i \begin{bmatrix} V(t) \\ \alpha(t) \\ q(t) \\ \theta(t) \end{bmatrix} + \bar{B}_1^i \begin{bmatrix} u_g(t) \\ w_g(t) \end{bmatrix}$$

$$+ \bar{B}_2^i \begin{bmatrix} \Delta C_D / C_{D_o} \\ \Delta C_{M_\alpha} / C_{M_{\alpha o}} \\ \Delta C_{M_\alpha} / C_{M_{\alpha o}} \end{bmatrix} + \bar{B}_3^i \delta_e(t) \tag{35}$$

The vehicle states $\bar{x}^i(t)$ are velocity $V(t)$ (in feet per second), angle of attack $\alpha(t)$ (radians), pitch rate $q(t)$ (radians per second), and pitch attitude $\theta(t)$ (radians). The control input $u^i(t)$ is the usual elevator control $\delta_e(t)$ (radians). The disturbance inputs $w_2^i(t) = [u_g(t), w_g(t)]^T$ for the $\mathcal{H}_2$ performance are associated with the longitudinal and transverse components of the atmospheric turbulence having (worst-case) white-noise spectra. The output performance variables $z_2^i(t)$ consist of a weighted combination of aircraft velocity, pitch attitude, and elevator control. In this design, we consider robustness to percent variations in the drag coefficient C_D and in the pitching moment coefficient to changes in the angle of attack C_{M_α}. The latter coefficient has a direct influence on the vehicle static stability. We model these real parameter uncertainties in the form of an uncertainty block $\Delta^i(s)$ having a specific diagonal structure from which we define the input vector $w_\infty^i(t)$ and the output vector $z_\infty^i(t)$. In this example, we treat the uncertainty block as unstructured and attempt to synthesize a controller that provides robustness to these normalized variations. State model matrices for the two design flight conditions are given below:

$$\bar{A}^1 = \begin{bmatrix} -0.00819 & -25.70839 & 0 & -32.17095 \\ -0.00019 & -1.27626 & 1.0000 & 0 \\ 0.00069 & 1.02176 & -2.40523 & 0 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix}$$

$$\bar{B}_1^1 = \begin{bmatrix} 0.00819 & 0.04621 \\ 0.00019 & 0.00229 \\ -0.00069 & -0.00183 \\ 0 & 0 \end{bmatrix}$$

$$\bar{B}_2^1 = \begin{bmatrix} -0.55854 & 0 & 0 \\ 0 & -0.27927 & 0 \\ 0 & 0.99911 & 20.99383 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\bar{B}_3^1 = \begin{bmatrix} -6.80939 \\ -0.14968 \\ -14.06111 \\ 0 \end{bmatrix}$$

$$\bar{C}_1^1 = \begin{bmatrix} 0.07071 & 0 & 0 & 0 \\ 0 & 0 & 0.31623 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \bar{D}_{13}^1 = \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix}$$

$$\bar{C}_2^1 = \begin{bmatrix} 0.01468 & 0 & 0 & 0 \\ 0 & 0.01468 & 0 & 0 \\ 0 & -0.16882 & 0 & 0 \end{bmatrix}, \qquad \bar{C}_3^1 = I$$

$$\bar{A}^2 = \begin{bmatrix} -0.01172 & -95.91071 & 0 & -32.11294 \\ -0.00011 & -1.87942 & 1.0000 & 0 \\ 0.00056 & -3.61627 & -3.44478 & 0 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix}$$

$$\bar{B}_1^2 = \begin{bmatrix} 0.01172 & 0.06607 \\ 0.00011 & 0.00129 \\ -0.00056 & 0.00249 \\ 0 & 0 \end{bmatrix}$$

$$\bar{B}_2^2 = \begin{bmatrix} -0.79850 & 0 & 0 \\ 0 & -0.39925 & 0 \\ 0 & 2.04570 & 78.46350 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\bar{B}_3^2 = \begin{bmatrix} -25.40405 \\ -0.22042 \\ -53.42460 \\ 0 \end{bmatrix}$$

$$\bar{C}_1^2 = \bar{C}_1^1, \bar{D}_{13}^2 = \bar{D}_{13}^1, \qquad \bar{C}_2^2 = \bar{C}_2^1, \bar{C}_3^2 = I$$

All other matrices are assumed to be zero. A preliminary analysis of the plants is given in Table 1. Both plant conditions are open-loop stable. The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of the respective open-loop subsystems are relatively large; in particular the $\mathcal{H}_\infty$ norms indicate that the open-loop systems are fairly sensitive to variation in the parameters C_D and C_{M_α}. The minimally achievable $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms shown in this table have been achieved for each plant condition separately by solving single-plant $\mathcal{H}_2$ and single-plant $\mathcal{H}_\infty$ design problems. These results indicate that one can improve the performance and robustness dramatically using a first-order feedback controller. Figures 1 and 2 show the design trade-offs involved in this problem. Figure 1 represents the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design for the first plant condition only, without taking into consideration the second design condition. The trade-off between performance and robustness is typical. Performance improvement implies invariably a deterioration in stability robustness and vice versa. For all the design points the achieved $\|T_\infty^1(C_o, s)\|_\infty$ saturates the γ bound specified. The best compromise between performance and robustness is at the point $\|T_\infty^1(C_o, s)\|_\infty \approx 0.1$ with a corresponding $\|T_2^1(C_o, s)\|_2 \approx 0.018$. Dramatic reduction in either robustness or performance can be achieved at the other design points. The corresponding "performance/robustness" characteristic for the second plant condition is not shown since the results would not be meaningful. Figure 2 shows the design trade-off resulting from taking into account both plant conditions during the design. The weighting factors α^i were chosen to be $\alpha^1 = \alpha^2 = 1$ (hence both $\mathcal{H}_2$ norms are weighted equally). This choice is justified as both plant conditions have roughly the same value for the minimally achievable $\mathcal{H}_2$ norm. The same value γ_{spec} was applied as an $\mathcal{H}_\infty$ constraint to both plant conditions. Hence this is only a two-dimensional example out of a generally four-dimensional surface where α^1 and α^2 as well as γ_{spec}^1 and γ_{spec}^2 may be chosen independently. In a mixed design for multiple will provide an actual constraint for

Table 1 Preliminary analysis

	Plant 1	Plant 2
Open-loop stability	Stable	Stable
$\|T_2^1(s)\|_2$ (open loop)	0.107	0.031
$\|T_\infty^1(s)\|_\infty$ (open loop)	23348.3	8013.3
Minimum achievable $\mathcal{H}_2$ norm[a]	0.032	0.0022
Minimum achievable $\mathcal{H}_\infty$ norm[a]	0.056	0.096

[a]Using a first-order controller.

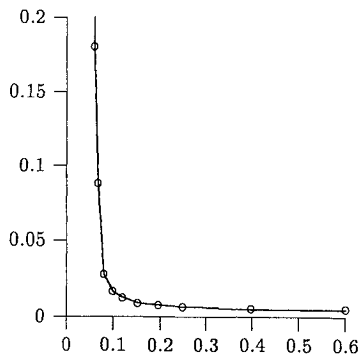

Fig. 1 Design considering only first plant condition.

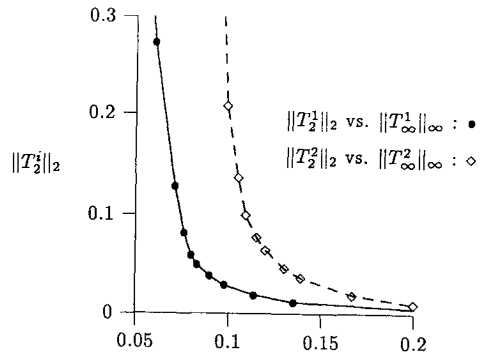

Fig. 2 Design considering both plant conditions.

only some of the n_p operating conditions, leaving the other plants unconstrained in terms of the robustness. In our example the resulting $\|T_\infty^1(C_o, s)\|_\infty$ was always below the specified γ_{spec} while $\|T_\infty^2(C_o, s)\|_\infty$ stayed on the specified robustness boundary for all the design points.

From Fig. 2 it follows that the best compromise between robustness and performance for the two given flight conditions is at the point where $\|T_2^1(C_o, s)\|_2 = 0.058$, $\|T_\infty^1(C_o, s)\|_\infty = 0.081$, $\|T_2^2(C_o, s)\|_2 = 0.077$, and $\|T_\infty^2(C_o, s)\|_\infty = 0.115$. This compromise is achieved with the following first-order controller:

$$A_c = [-8.55194]$$

$$B_c = [-53.16826 \quad -5.65807 \quad -19.24661 \quad -5.53476]$$

$$C_c = [0.07340]$$

$$D_c = [-0.83757 \quad 15.44903 \quad 1.17896 \quad 8.66218]$$

VII. Conclusions

In this paper we developed a new approach for addressing multimodel mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problems. This approach is based upon the use of a time-domain scalar cost function that includes the $\mathcal{H}_\infty$ constraints as a penalty function. Using the proposed formulation, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design becomes an unconstrained optimization problem that provides the optimal $\mathcal{H}_2$ performance measure subject to the given $\mathcal{H}_\infty$ bounds. The resulting optimization problem is smooth, and hence standard gradient-based methods can be used. The algorithmic treatment in a gradient-based finite-time setting allows for handling initially destabilizing controllers. Unlike previous approaches, this formulation minimizes the true $\mathcal{H}_2$ cost rather than an upper bound, and it does not require either identical inputs or outputs in the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problem formulation. The controller structure is arbitrary, allowing designers to consider a wide class of controllers including proper and strictly proper controllers with fixed order and fixed structure.

The only drawback of this formulation is given by the size of the optimization problem. This size depends both on the structure selected for the controller and on the order of the open-loop plants. It becomes increasingly complex for higher order systems and controllers. Thus inclusion of robust $\mathcal{H}_\infty$ stability into the nominal $\mathcal{H}_2$ framework entails increased computational complexity.

Finally, by suitably modifying the overall objective function, the approach can be easily extended to encompass a wide range of constraints, as long as these constraints can be expressed in terms of matrix inequalities.

Appendix: Gradient Computation

Assume a closed-loop system $\Sigma_{\infty,\text{cl}}^i(C_o)$ as in Eq. (6) and let

$$\hat{A}^i = A^i + B_2^i(\bar{R}^i)^{-1}D_{22}^{iT}C_2^i$$

$$\hat{B}^i = B_2^i(\bar{R}^i)^{-1}B_2^{iT}$$

$$\hat{C}^i = C_2^{iT}\left[D_{22}^i(\bar{R}^i)^{-1}D_{22}^{iT} + I\right]C_2^i$$

with $\bar{R}^i = [(\gamma^i)^2 I - D_{22}^{iT}D_{22}^i]$. Note that the matrices $\hat{A}^i$, $\hat{B}^i$, and $\hat{C}^i$ are functions of C_o. As proposed, we use the Cholesky factorization of $X^i = Q^{iT}Q^i$, where Q^i are upper triangular matrices. The corresponding $\text{ARI}^i(C_o, X^i) = \text{ARI}^i(C_o, Q^{iT}Q^i)$ for the ith plant condition is of the form

$$\text{ARI}^i(C_o, Q^i) = \hat{A}^{iT}Q^{iT}Q^i + Q^{iT}Q^i\hat{A}^i + Q^{iT}Q^i\hat{B}^iQ^{iT}Q^i + \hat{C}^i \tag{A1}$$

Using the fact that $\text{Tr}(TU) = \text{Tr}(UT)$ for compatible matrices U and T and a power series expansion of $\exp\{\text{ARI}^i[C_o, (Q^i + \epsilon\,\Delta Q^i)]t_f\}$, it can be shown that

$$J_\infty^i\left[C_o, (Q^i + \epsilon\,\Delta Q^i), t_f\right] - J_\infty^i\left(C_o, Q^i, t_f\right) = 2\epsilon t_f \,\text{Tr}\left\{\mathcal{F}\left(C_o, Q^i\right)Q^{iT}\,\Delta Q^i\right\} \tag{A2}$$

where

$$\mathcal{F}\left(C_o, Q^i\right) = (\hat{A}^i + \hat{B}^iQ^{iT}Q^i)\exp\left[\text{ARI}^i\left(C_o, Q^i\right)t_f\right] + \exp\left[\text{ARI}^i\left(C_o, Q^i\right)t_f\right](\hat{A}^i + \hat{B}^iQ^{iT}Q^i)^T \tag{A3}$$

In this derivation all higher order terms in ϵ have been discarded. Hence, applying Kleinman's lemma (see Refs. 17 and 22), the first derivative of $J_\infty^i(C_o, Q^i, t_f)$ with respect to Q^i is given by

$$\frac{\partial\,\text{Tr}\left\{\exp\left[\text{ARI}^i\left(C_o, Q^i\right)t_f\right]\right\}}{\partial Q^i} = 2t_f Q^i\left[\mathcal{F}\left(C_o, Q^i\right)\right]^T \tag{A4}$$

In order to obtain the gradient with respect to C_o, explicit expressions of the closed-loop system matrices A^i, B_2^i, C_2^i, and D_{22}^i in terms of C_o are required. These expressions are given by

$$A^i = \tilde{A}^i + \tilde{B}_3^iC_o\tilde{C}_3^i$$

$$B_2^i = \tilde{B}_2^i + \tilde{B}_3^iC_o\tilde{D}_{32}^i$$

$$C_2^i = \tilde{C}_2^i + \tilde{D}_{23}^iC_o\tilde{C}_3^i$$

$$D_{22}^i = \tilde{D}_{22}^i + \tilde{D}_{23}^iC_o\tilde{D}_{32}^i$$

where

$$\tilde{A}^i = \begin{bmatrix} \bar{A}^i & 0 \\ 0 & 0 \end{bmatrix}, \qquad \tilde{B}_2^i = \begin{bmatrix} \bar{B}_2 \\ 0 \end{bmatrix}, \qquad \tilde{B}_3^i = \begin{bmatrix} \bar{B}_3 & 0 \\ 0 & I \end{bmatrix}$$

$$\tilde{C}_2^i = [\, \bar{C}_2 \quad 0 \,], \qquad \tilde{C}_3^i = \begin{bmatrix} \bar{C}_3 & 0 \\ 0 & I \end{bmatrix}$$

$$\tilde{D}_{22}^i = \bar{D}_{22}, \qquad \tilde{D}_{33}^i = [\, \bar{D}_{23} \quad 0 \,], \qquad \tilde{D}_{32}^i = \begin{bmatrix} \bar{D}_{32} \\ 0 \end{bmatrix}$$

By using the same technique, we obtain

$$\frac{\partial \operatorname{Tr}\{\exp[\mathrm{ARI}^i(C_o, Q^i)t_f]\}}{\partial C_o} = 2t_f[\mathcal{G}(C_o, Q^i)]^T \tag{A5}$$

where

$$\mathcal{G}(C_o, Q^i) = \{\tilde{D}_{23}^i(\bar{R}^i)^{-1}(D_{22}^{iT}C_2^i + B_2^i Q^{iT} Q^i)\}$$
$$\times \exp[\mathrm{ARI}^i(C_o, Q^i)t_f]\Big\{(D_{22}^{iT}C_2^i + B_2^i Q^{iT} Q^i)^T$$
$$\times D_{22}^{iT}(\bar{S}^i)^{-1}\tilde{D}_{23}^i + (\tilde{D}_{23}^{iT}C_2^i + B_2^i Q^{iT} Q^i)^T\Big\} \tag{A6}$$

The overall gradient of $J_\infty^i(C_o, X^i = Q^{iT}Q^i, t_f)$ follows from the summation of the individual gradients for each plant condition. Note that only matrix exponentials and matrix multiplications are needed to compute the gradients. Expressions for the derivative of $J_2(C_o, t_f)$ and the steady-state $\mathcal{H}_2$ costs $J_2(C_o, t_f \to \infty) = \|T_2^i(C_o)\|_2^2$ with respect to C_o can be found in Refs. 21 and 14, respectively.

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